MATH 161 - Fall 2013

Lecture 2

LECTURE 2 08/29/13 (1)

1.2 CATALOG OF ESSENTIAL FUNCTION

- Linear function : $y = f(x) = mx + b$, $m \equiv$ slope of the function, $b \equiv$ y-intercept.

  $\downarrow$

  rate of change of y with respect to x

- Polynomial : A function P is called a polynomial if

  $P(x) = a_n x^n + \dots + a_1 x^1 + a_0$, where n is a non-negative integer

  and the numbers $a_0, a_1, \dots, a_n$ are constants called coefficients.

Domain of a polynomial is $\mathbb{R} = (-\infty, \infty)$

If $a_n \neq 0$, the degree of polynomial is n.

Ex $P(x) = 2x^6 + x^5 + 3x + \sqrt{7}$

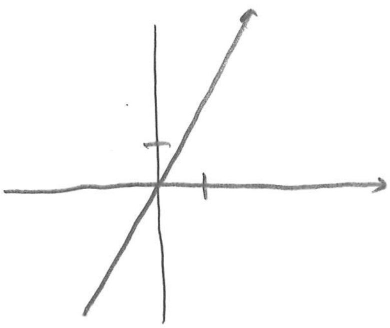
Degree 1 : Linear function

Degree 2 : Quadratic function

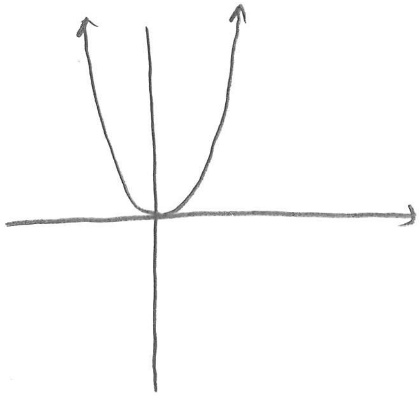
Degree 3 : Cubic function.

- Graphs of power functions : $f(x) = x^a$, where a is constant

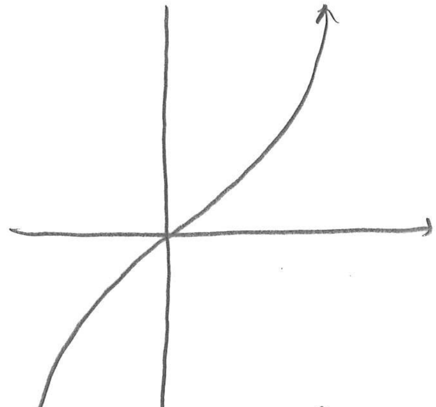
Case 1 $a = 1, 2, 3, \ldots$ [positive integer]



$y = x$



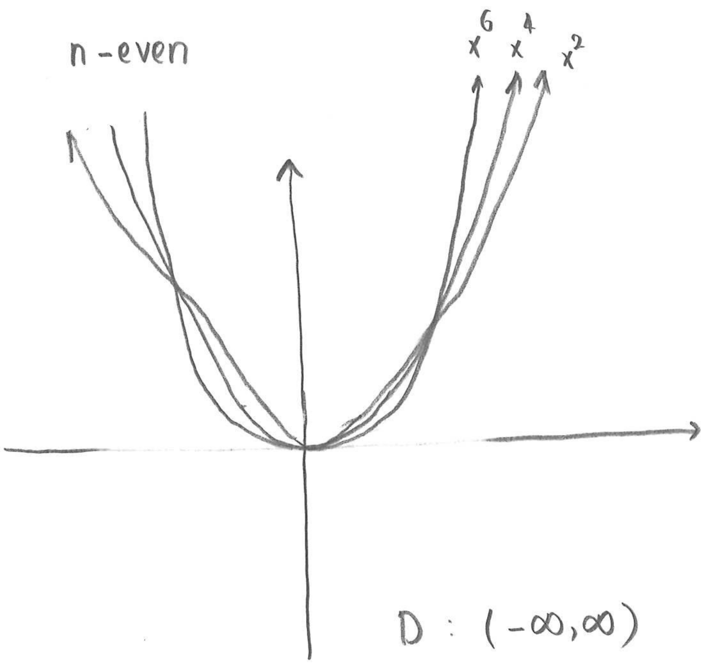
$y = x^2$



$y = x^3$

In general,

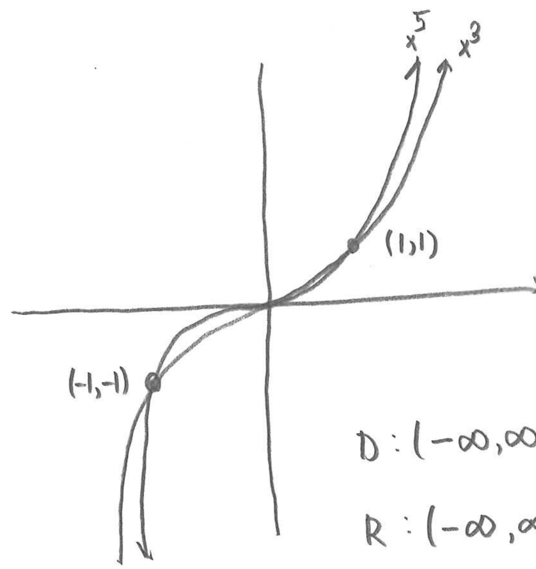
n - even



D : $(-\infty, \infty)$

R : $[0, \infty)$

n - odd



D : $(-\infty, \infty)$

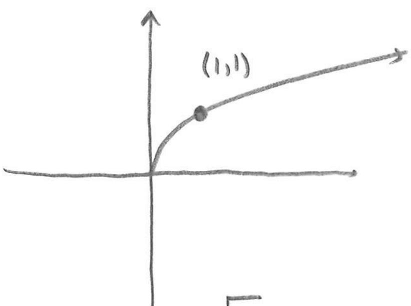
R : $(-\infty, \infty)$

Case 2 $a = \frac{1}{n}$, n is a positive integer

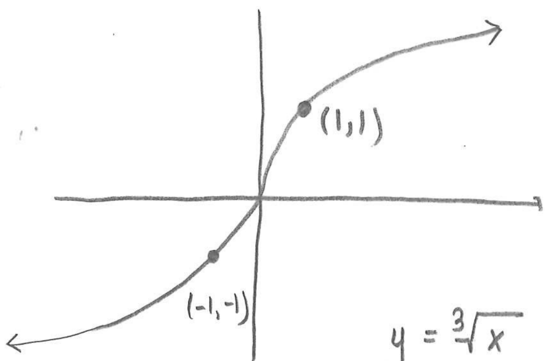
$f(x) = x^{1/n} = \sqrt[n]{x}$

For n-even, domain is $[0, \infty)$

For n-odd, domain is $(-\infty, \infty)$

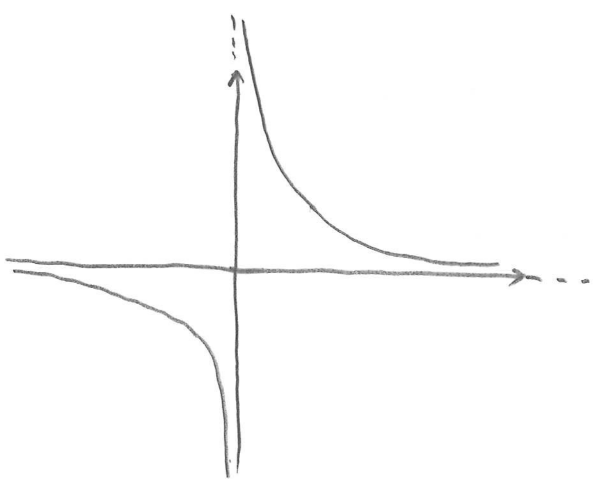


$\sqrt{x} = y$



$y = \sqrt[3]{x}$

<u>Case 3</u> $a = -1$, $y = \frac{1}{x}$



RATIONAL FUNCTIONS

These are functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where P, Q are polynomials.

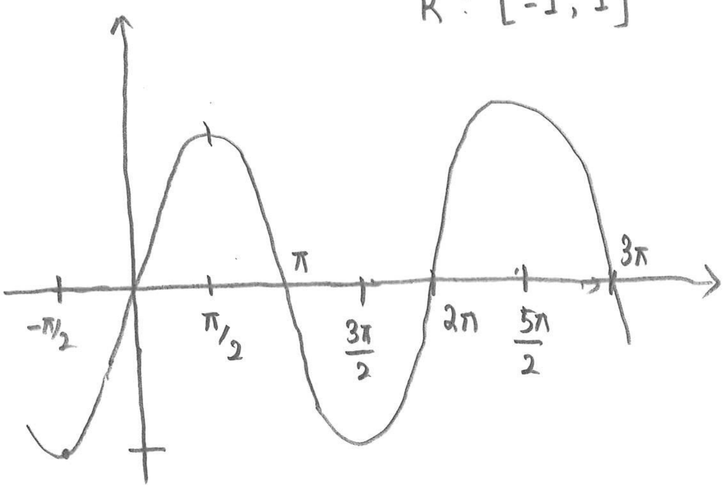
Domain of $f(x)$ consists of all x such that $Q(x) \neq 0$.

TRIGONOMETRIC FUNCTIONS

$f(x) = \sin x$

$D: (-\infty, \infty)$

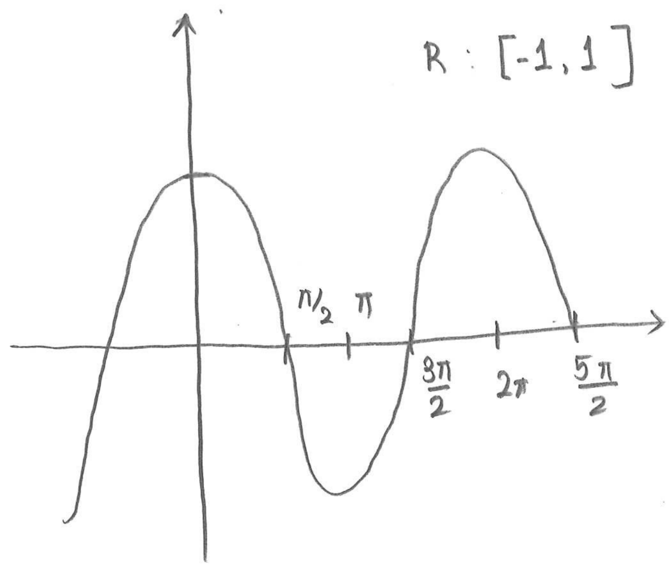
$R: [-1, 1]$

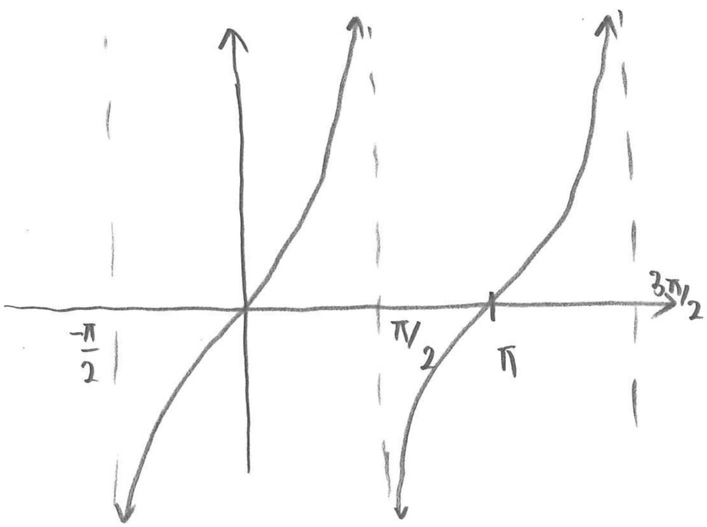


$f(x) = \cos x$

$D: (-\infty, \infty)$

$R: [-1, 1]$



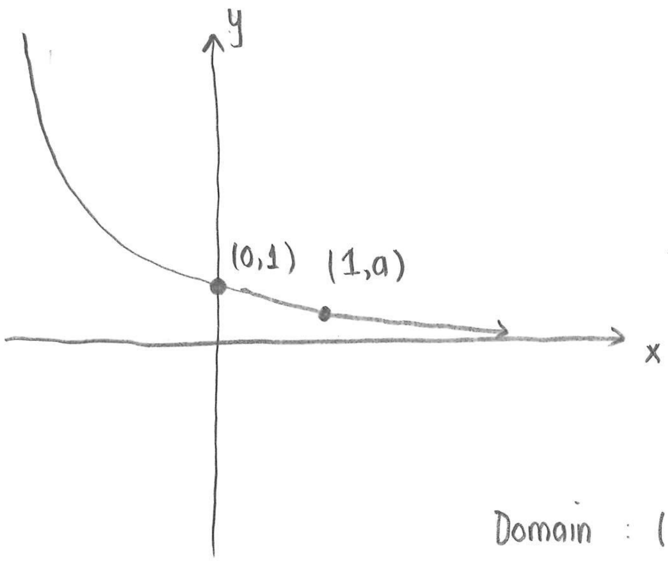


$D: x = \frac{n\pi}{2}$, $n = \ldots, -3, -1, 1, 3, \ldots$

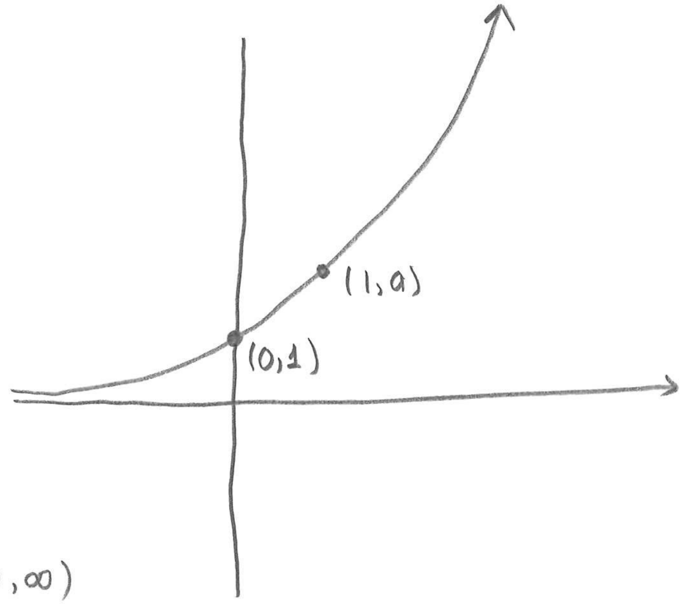
$R = (-\infty, \infty)$

$f(x) = a^x$

$0 < a < 1$



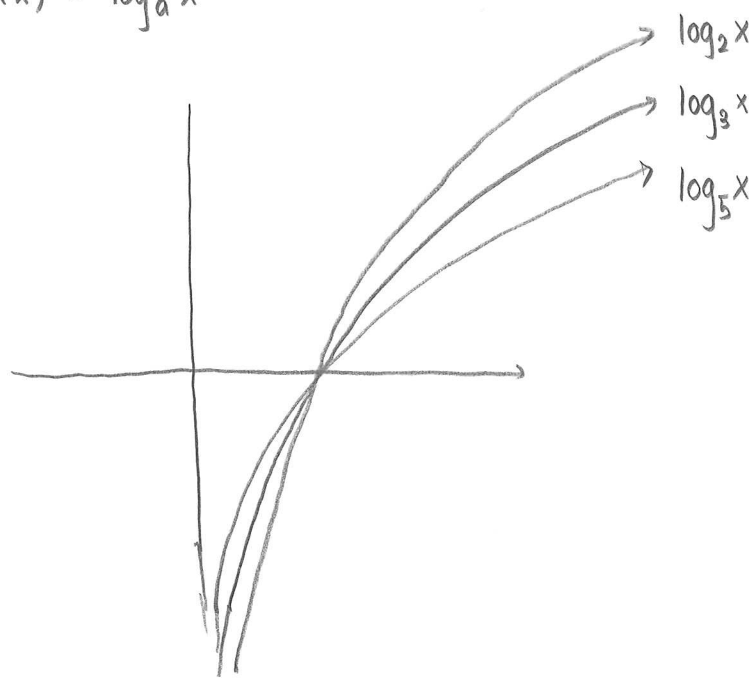
$a > 1$



Domain : $(-\infty, \infty)$

Range : $(0, \infty)$

$f(x) = \log_a x$



Domain : $(0, \infty)$

Range : $(-\infty, \infty)$

TRANSLATIONS OF FUNCTIONS

VERTICAL AND HORIZONTAL SHIFTS

Let $c > 0$. Then the graph of

$y = f(x) + c$ is obtained by shifting the graph of $y = f(x)$, c units upwards.

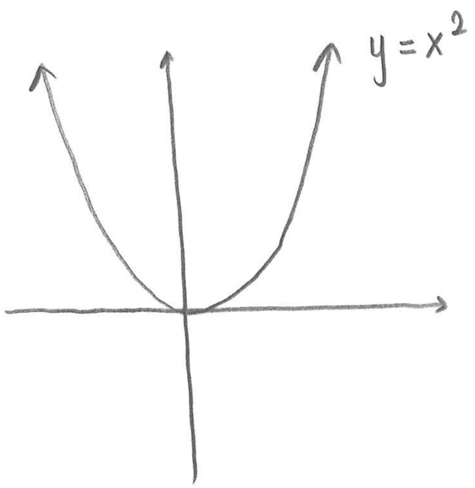
$y = f(x) - c$ " " " " " " " " , c units downwards

$y = f(x-c)$ " " " " " " " " , c units right

$y = f(x+c)$, c units left

<u>Ex</u> Let $y = \underbrace{(x-3)^2 + 2}$

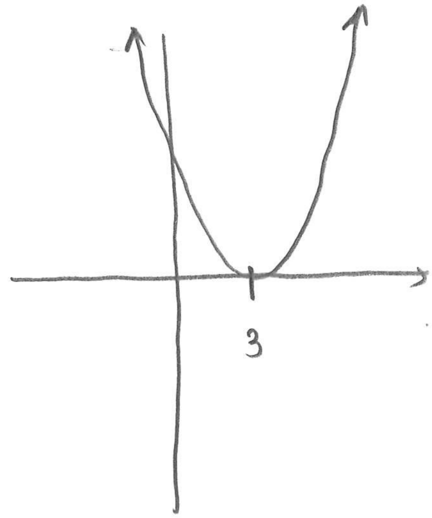
parent function $y = x^2$



SHIFT RIGHT

$\Longrightarrow$

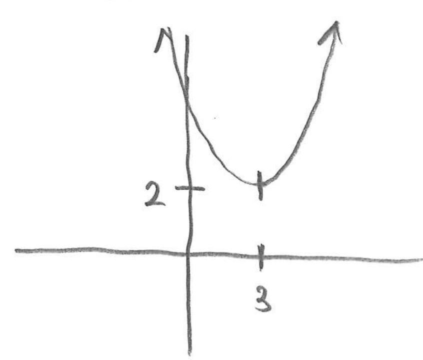
3 units



SHIFT UP

$\Longrightarrow$

2 UNITS



VERTICAL AND HORIZONTAL STRETCHING, SHRINKING, AND REFLECTING

Let $c > 1$.

$y = cf(x)$, stretch $y = f(x)$ vertically by a factor of c.

$y = \frac{1}{c} f(x)$, shrink $y = f(x)$ vertically by a factor of c

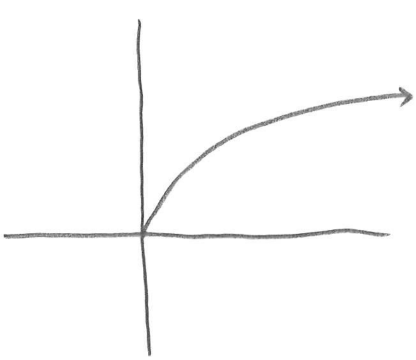
$y = f(cx)$, shrink $y = f(x)$ horizontally by a factor of c

$y = f\left(\frac{x}{c}\right)$, stretch $y = f(x)$ " " " "

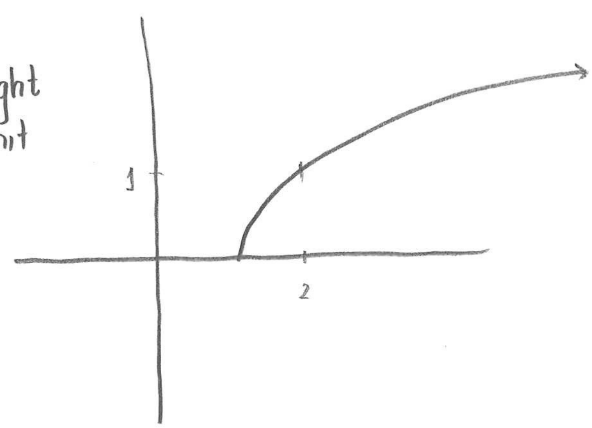
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

$y = f(-x)$, " " " " " about the y-axis:

Ex $y = \underbrace{-2\sqrt{(x-1)} + 1}_{\text{parent function } y = \sqrt{x}}$

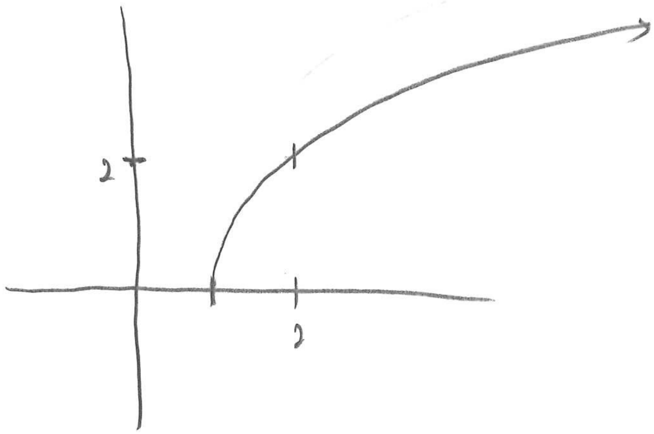


shift right by 1-unit $\Longrightarrow$



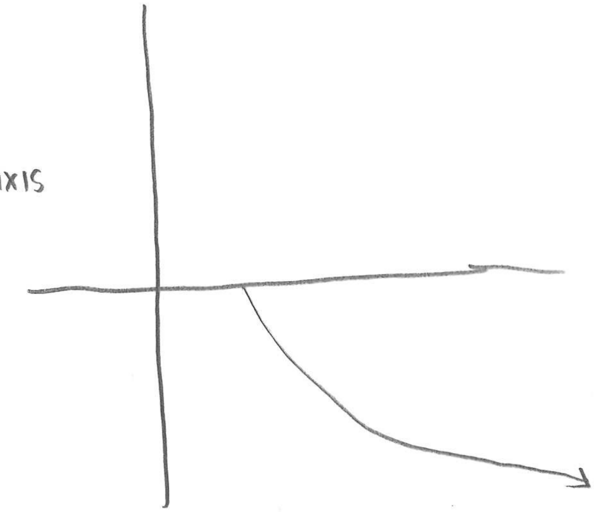
$\Longrightarrow$

$\Rightarrow$



$2\sqrt{x-1}$

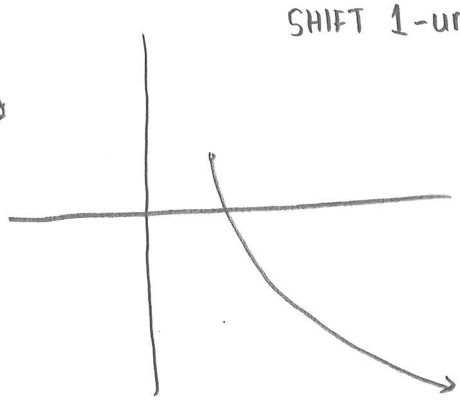
Reflect about x-axis $\Rightarrow$



$-2\sqrt{x-1}$

SHIFT 1-unit up

$\Rightarrow$



$-2\sqrt{x-1}+1$

FUNCTION OPERATIONS

Given two functions f and g, they can be combined to form new functions

$f+g$, $f-g$, fg, $\frac{f}{g}$.

$(f+g)(x) = f(x)+g(x)$, $(f-g)(x) = f(x)-g(x)$, $(fg)(x) = f(x).g(x)$.

So what about their domains ?

If A is the domain of f, B is the domain of g, the domain of

$f+g$ is $A \cap B$.

Same for $f-g$ and fg.

Ex $f(x) = \frac{1}{x-1}$ and $g(x) = \sqrt{x+3}$

$$\Downarrow \qquad\qquad \Downarrow$$

$$D : x \neq 1 \quad \cap \quad D : x \geqslant -3$$

$$[-3, 1) \cup (1, \infty)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

In this case we can't divide by 0. So we have an added restriction that $g(x) \neq 0$

So domain of $\frac{f}{g}$ is $\{x \mid x \in A \cap B \ \& \ g(x) \neq 0\}$

FUNCTION COMPOSITION

Refers to the combining of functions in a manner where the output from one function becomes input for the next function.

NOTATION $f \circ g(x)$ Read f of $g(x)$

Take $g(x)$ and plug it into $f(x)$. $f \circ g(x) = f[g(x)]$

Ex $f(x) = x^2$ and $g(x) = x - 3$. Find $f \circ g(x)$ and $g \circ f(x)$.

$$f \circ g(x) = f[g(x)] = f(x-3)$$

* Aside, if you wanted to find $f(3)$, what would you do ??

You would plug in 3 for x, so $f(3) = 3^2 = 9$

Similarly, to find $f(x-3)$, we plug in $(x-3)$ instead of x, so

$f(x-3) = (x-3)^2$.

Similarly,

$$(g \circ f)(x) = g[f(x)] = g(x^2) = x^2 - 3$$

IMPORTANT $g \circ f \neq f \circ g$

Remark What is the domain of $f \circ g$??

The domain of $f \circ g$ is all x in the domain of g s.t $g(x)$ is the domain of f.

In other words, $(f \circ g)(x)$ is defined when both $g(x)$ and $f \circ g(x)$ are defined.

Example

$f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$

Domain of f : $x \geqslant 0$ or $[0, \infty)$

Domain of g : $2 - x \geqslant 0 \Rightarrow 2 \geqslant x$ or $(-\infty, 2]$

a) $f \circ g(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$

Domain of $\sqrt[4]{2-x}$ is $2-x \geq 0$ or $2 \geq x$ or $(-\infty, 2]$

Domain of g is $(-\infty, 2]$

Hence domain of $f \circ g$ is $(-\infty, 2]$

b) $g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

Domain of $\sqrt{2-\sqrt{x}}$

STEP 1 Since we have $\sqrt{x}$ inside, $x \geq 0$

STEP 2 $2-\sqrt{x} \geq 0$ or $\sqrt{x} \leq 2 \Rightarrow x \leq 4$ $\left[\begin{array}{c} \text{If } 0 \leq a \leq b \\ \Downarrow \\ a^2 \leq b^2 \end{array} \right]$

Then intersection $[0, 4]$ is the domain of $\sqrt{2-\sqrt{x}}$

Domain of f is $[0, \infty)$

Hence domain of $g \circ f$ is $[0, 4]$

ⓒ $g \circ g(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$

Domain of $\sqrt{2-\sqrt{2-x}}$

STEP 1 $\quad 2-x \geqslant 0$ or $2 \geqslant x \Rightarrow x \in (-\infty, 2]$

STEP 2 $\quad 2-\sqrt{2-x} \geqslant 0 \Rightarrow 2 \geqslant \sqrt{2-x} \Rightarrow 4 \geqslant 2-x$

$\Rightarrow 4+x \geqslant 2 \Rightarrow x \geqslant -2 \Rightarrow x \in [-2, \infty)$

Then the intersection $[-2,2]$ is the domain of $\sqrt{2-\sqrt{2-x}}$

Domain of $g(x)$ is $(-\infty, 2]$

Hence domain of $g \circ g$ is $[-2,2]$.

So previous example was build complicated function from simpler ones.

But we also need to be able to decompose complicated functions into simpler ones.

Ex $F(x) = \cos^2(x+9)$. Find functions f, g, h s.t $F = f \circ g \circ h$

$F(x) = [\cos(x+9)]^2$

So what we are doing is adding 9, the taking cosine of the result and finally squaring

So $h(x) = x+9$, $g(x) = \cos x$ $f(x) = x^2$

Then $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) = [\cos(x+9)]^2$

Ex $F(x) = (x+3)^2 + 6$

Add 3, square and Add 6

$h(x) = x+3$, $g(x) = x^2$, $f(x) = x+6$

Ex $(x^2+3)^2$

So square, Add 3, square

$h(x) = x^2$

$g(x) = x+3$

$f(x) = x^2$